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Well-posedness of mixed variational inequalities, inclusion problems and fixed point problems

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Abstract We generalize the concept of well-posedness to a mixed variational inequality and give some characterizations of its well-posedness. Under suitable conditions, we prove that the well-posedness of a mixed variational inequality is equivalent to the well-posedness of a corresponding inclusion problem. We also discuss the relations between the well-posedness of a mixed variational inequality and the well-posedness of a fixed point problem. Finally, we derive some conditions under which a mixed variational inequality is well-posed.

Keywords Mixed variational inequality · Inclusion problem · Fixed point problem · Well-posedness

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1 Introduction

Tykhonov [26] first introduced the concept of well-posedness for a minimization problem, which has been known as Tykhonov well-posedness. Roughly speaking, the Tykhonov well-posedness of a minimization problem means the existence and uniqueness of minimizers, and the convergence of every minimizing sequence toward the unique minimizer. In many

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practical situations, there are more than one minimizers for a minimization problem. In this case, the concept of Tykhonov well-posedness in the generalized sense was introduced, which means the existence of minimizers and the convergence of some subsequence of every minimizing sequence toward a minimizer. Clearly, the concept of well-posedness is motivated by the numerical methods producing optimizing sequences. Because of its importance in optimization problems, various concepts of well-posedness have been introduced and studied for minimization problems in past decades. For details, we refer the readers to [1,6,10,18,24,26,29,30] and the references therein.

In recent years, the concept of well-posedness has been generalized to other contexts: variational inequality problems [5,8,15–18], saddle point problems [4], Nash equilibrium problems [17,19–23,25], inclusion problems [13,14], and fixed point problems [13,14,27]. Concerning the well-posedness of a given variational problem, it is interesting and important to establish its metric characterization, to find conditions under which the problem is well-posed, to investgate its links with the well-posedness of other related problems. Some metric characterizations of various well-posedness were established for minimization problems [6], variational inequalities [5,8,15,16] and Nash equilibrium problems [22]. For the well-posedness conditions of various variational problems, we refer the readers to [5,6,8,15,16,23,25]. The relations between the well-posedness of variational inequalities and the well-posedness of minimization problems were discussed in [5,16,18]. Lemaire [13] discussed the relations among the well-posedness of minimization problems, inclusion problems and fixed point problems. Recently, Lemaire et al. [14] further extended the result in ref. [13] by considering perturbations.

Motivated by the afore-mentioned works, in this paper we investigate the well-posedness of a mixed variational inequality which includes as a special case the classical variational inequality. We give some metric characterizations of its well-posedness and establish the links with the well-posedness of inclusion problems and fixed point problems. Finally, we prove that under suitable conditions, the well-posedness of the mixed variational inequality is equivalent to the existence and uniqueness of its solutions, and the well-posedness in the generalized sense is equivalent to the existence of solutions.

2 Preliminaries

Let H be a real Hilbert space, $F\colon H \to H$ be a mapping and $\varphi\colon H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Denote by $dom\,\varphi$ the domain of φ, i.e.,

$$dom\,\varphi = \{x \in H : \varphi(x) < +\infty\}.$$

Consider the following mixed variational inequality associated with (F, φ):

$$\text{MVI}(F,\varphi): \qquad \text{find } x \in H \text{ such that } \langle F(x), x-y\rangle + \varphi(x) - \varphi(y) \le 0, \quad \forall y \in H,$$

which has been studied intensively (see, e.g., [2,7,9,28]). When $\varphi = \delta_K$, MVI(F,φ) reduces to the classical variational inequality:

$$\text{VI}(F,K): \qquad \text{find } x \in K \text{ such that } \langle F(x), x-y\rangle \le 0, \quad \forall y \in K,$$

where δ_K denotes the indicator functional of a convex subset K of H. Denote by $\partial\varphi$ and $\partial_\epsilon\varphi$ the subdifferential and ϵ-subdifferential of φ respectively, i.e.,

$$\partial\varphi(x) = \{x^* \in H : \varphi(y) - \varphi(x) \ge \langle x^*, y-x\rangle, \forall y \in H\}, \quad \forall x \in dom\,\varphi,$$
$$\partial_\epsilon\varphi(x) = \{x^* \in H : \varphi(y) - \varphi(x) \ge \langle x^*, y-x\rangle - \epsilon, \forall y \in H\}, \quad \forall x \in dom\,\varphi.$$

It is known that $\partial_\epsilon\varphi(x) \supset \partial\varphi(x) \neq \emptyset$ for all $x \in dom\varphi$ and for all $\epsilon > 0$. In terms of $\partial\varphi$, MVI(F, φ) is equivalent to the following inclusion problem associated with $F + \partial\varphi$:

$$\text{IP}(F + \partial\varphi): \qquad \text{find } x \in H \text{ such that } 0 \in F(x) + \partial\varphi(x).$$

The resolvent operator of $\partial\varphi$ is defined by

$$J_\varphi^\lambda(x) = (I + \lambda\partial\varphi)^{-1}(x), \quad \forall x \in H,$$

which is well-defined, single-valued and nonexpansive, where $\lambda > 0$ is a constant. Recall that a mapping $T : H \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in H$. In terms of J_φ^λ, MVI$(F, \partial\varphi)$ is also equivalent to the following fixed point problem:

$$\text{FP}(J_\varphi^\lambda(I - \lambda F)): \quad \text{find } x \in H \text{ such that } x = J_\varphi^\lambda(I - \lambda F)(x).$$

Summarizing the above results, we have the following lemma:

Lemma 2.1 (See, e.g., [2,9,28]) *Let $F: H \to H$ be a mapping and $\varphi: H \to R \cup \{+\infty\}$ be a proper, convex and lower semicontinuous functional. Then the following conclusions are equivalent:*

(i) *x solves* MVI(F, φ)*;*

(ii) *x solves IP$(F + \partial\varphi)$;*

(iii) *x solves FP$(J_\varphi^\lambda(I - \lambda F))$, where $\lambda > 0$ is a constant.*

In the sequel we recall some concepts.

Definition 2.1 A mapping $F: H \to H$ is said to be monotone if

$$\langle F(x) - F(y), x - y\rangle \geq 0, \quad \forall x, y \in H.$$

Definition 2.2 A mapping $F: H \to H$ is said to be hemicontinuous if for any $x, y \in H$, the function $t \mapsto \langle F(x + t(y - x)), y - x\rangle$ from $[0, 1]$ into R is continuous at 0_+.

Clearly, the continuity implies the hemicontinuity, but the converse is not true in general.

Definition 2.3 A mapping $F: H \to H$ is said to be uniformly continuous if for any neighborhood V of 0, there exists a neighborhood U of 0 such that $F(x) - F(y) \in V$ for all $x, y \in U$. Obviously, the uniform continuity implies the hemicontinuity.

Lemma 2.2 (See, e.g. [2,9,28]) *Let $F: H \to H$ be monotone and hemicontinuous, $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous, and $x \in V$ a give point. Then*

$$\langle F(x), x - y\rangle + \varphi(x) - \varphi(y) \leq 0, \quad \forall y \in H$$

if and only if

$$\langle F(y), x - y\rangle + \varphi(x) - \varphi(y) \leq 0, \quad \forall y \in H.$$

Definition 2.4 (See [12]) Let A be a nonempty subset of H. The measure of noncompactness μ of the set A is defined by

$$\mu(A) = \inf\{\epsilon > 0: A \subset \cup_{i=1}^n A_i, \ \text{diam } \mathrm{A_i} < \epsilon, \ \mathrm{i} = 1, 2, \cdots, n\},$$

where diam means the diameter of a set.

Definition 2.5 Let A, B be nonempty subsets of H. The Hausdorff metric $\mathcal{H}(\cdot,\cdot)$ between A and B is defined by

$$\mathcal{H}(A, B) = \max\{e(A, B), e(B, A)\},$$

where $e(A, B) = \sup_{a \in A} d(a, B)$ with $d(a, B) = \inf_{b \in B} \|a - b\|$. Let $\{A_n\}$ be a sequence of nonempty subsets of H. We say that A_n converges to A in the sense of Hausdorff metric if $\mathcal{H}(A_n, A) \to 0$. It is easy to see that $e(A_n, A) \to 0$ if and only if $d(a_n, A) \to 0$ for all selection $a_n \in A_n$. For more details on this topic, we refer the readers to [11,12].

3 Well-posedness and metric characterization

In this section we introduce some concepts of well-posedness of the mixed variational inequality and establish their metric characterizations. Let $\alpha \geq 0$ be a given number and let H, F, φ be defined as in the previous section.

Definition 3.1 A sequence $\{x_n\} \subset H$ is called an α-approximating sequence for MVI(F, φ) if there exists a sequence $\{\epsilon_n\}$ of non-negative numbers with $\epsilon_n \to 0$ such that

$$x_n \in dom\ \varphi, \quad \langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in H, \forall n \in N.$$

If $\alpha_1 > \alpha_2 \geq 0$, then every α_2-approximating sequence is α_1-approximating. When $\alpha = 0$, we say that $\{x_n\}$ is approximating for MVI(F, φ).

Definition 3.2 We say that MVI(F, φ) is strongly (resp. weakly) α-well-posed if MVI(F, φ) has a unique solution and every α-approximating sequence converges strongly (resp. weakly) to the unique solution. In the sequel, strong (resp. weak) 0-well-posedness is always called as strong (resp. weak) well-posedness. If $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) α_1-well-posedness implies strong (resp. weak) α_2-well-posedness.

Remark 3.1 When $\varphi = \delta_K$, Definition 3.2 reduces to the definition of strong (resp. weak) α-well-posedness for the classical variational inequality. For details, we refer the readers to [5,16,17] and the references therein.

Definition 3.3 We say that MVI(F, φ) is strongly (resp. weakly) α-well-posed in the generalized sense if MVI(F, φ) has a nonempty solution set S and every α-approximating sequence has a subsequence which converges strongly (resp. weakly) to some point of S. When $\alpha = 0$, we say that MVI(F, φ) is strongly (resp. weakly) well-posed in the generalized sense. Clearly, if $\alpha_1 > \alpha_2 \geq 0$, then strong (resp. weak) α_1-well-posedness in the generalized sense implies strong (resp. weak) α_2-well-posedness in the generalized sense.

Remark 3.2 When $\varphi = \delta_K$, Definition 3.3 reduces to the definition of strongly (weakly) α-well-posedness in the generalized sense for the classical variational inequality. For details, we refer readers to [5,16,17] and the references therein.

The α-approximating solution set of MVI(F, φ) is defined by

$$\Omega_\alpha(\epsilon) = \{x \in H : \langle F(x), x - y\rangle + \varphi(x) - \varphi(y) \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \forall y \in H\}, \quad \forall \epsilon \geq 0.$$

Now we give a metric characterization of strong α-well-posedness for MVI(F, φ).

Theorem 3.1 *Let $F: H \to H$ be hemicontinuous and monotone and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. Then* MVI(F, φ) *is strongly α-well-posed if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \quad \text{and} \quad \text{diam}\, \Omega_\alpha(\epsilon) \to 0 \quad \text{as } \epsilon \to 0. \tag{1}$$

Proof Suppose that MVI(F, φ) is strongly α-well-posed. Then MVI(F, φ) has a unique solution which belongs to $\Omega_\alpha(\epsilon)$ for all $\epsilon > 0$. If diam $\Omega_\alpha(\epsilon) \not\to 0$ as $\epsilon \to 0$, then there exist constant $l > 0$ and sequences $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$, and $\{u_n\}, \{v_n\}$ with $u_n, v_n \in \Omega_\alpha(\epsilon_n)$ such that

$$\|u_n - v_n\| > l, \quad \forall n \in N. \tag{2}$$

Since $u_n, v_n \in \Omega_\alpha(\epsilon_n)$, both $\{u_n\}$ and $\{v_n\}$ are α-approximating sequences for MVI(F, φ). So they have to converge strongly to the unique solution of MVI(F, φ), a contradiction to (2).

Conversely, suppose that condition (1) holds. Let $\{x_n\} \subset H$ be an α-approximating sequence for MVI(F, φ). Then there exists a sequence $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$ such that

$$x_n \in dom\, \varphi, \quad \langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in H, \forall n \in N. \tag{3}$$

This yields that $x_n \in \Omega_\alpha(\epsilon_n)$. From (1), we know that $\{x_n\}$ is a Cauchy sequence and so it converges strongly to a point $\bar{x} \in H$. Since F is monotone and φ is lower semicontinuous, it follows from (3) that

$$\begin{aligned}
&\langle F(y), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \\
&\quad \leq \liminf_{n\to\infty}\{\langle F(y), x_n - y\rangle + \varphi(x_n) - \varphi(y)\} \\
&\quad \leq \liminf_{n\to\infty}\{\langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y)\} \\
&\quad \leq \liminf_{n\to\infty}\{\frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n\} \\
&\quad = \frac{\alpha}{2}\|\bar{x} - y\|^2, \quad \forall y \in H.
\end{aligned}$$

For any $y \in H$, let $y_t = (1-t)\bar{x} + ty$, $t \in [0, 1]$. Then

$$\langle F(y_t), \bar{x} - y_t\rangle + \varphi(\bar{x}) - \varphi(y_t) \leq \frac{\alpha}{2}\|\bar{x} - y_t\|^2.$$

Since φ is convex,

$$\langle F(y_t), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \leq \frac{t\alpha}{2}\|\bar{x} - y\|^2.$$

Letting $t \to 0$ in the above inequality, we get

$$\langle F(y), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \leq 0, \quad \forall y \in H.$$

By Lemma 2.2, $\bar{x}$ solves MVI(F, φ).

To complete the proof, we need only to prove that MVI(F, φ) has a unique solution. Assume by contradiction that MVI(F, φ) has two distinct solution x_1 and x_2. Then it is easy to see that $x_1, x_2 \in \Omega_\alpha(\epsilon)$ for all $\epsilon > 0$ and

$$0 < \|x_1 - x_2\| \leq \text{diam}\, \Omega_\alpha(\epsilon) \to 0,$$

a contradiction to (1). □

Remark 3.3 Theorem 3.1 generalizes Proposition 2.2 of [5].

In terms of noncompact measure, we have the following analogous metric characterization of strong α-well-posedness in the generalized sense.

Theorem 3.2 *Let $F: H \to H$ be such that the functional $x \mapsto \langle F(x), x - y\rangle$ is lower semicontinuous for all $y \in H$, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. Then* MVI(F, φ) *is strongly α-well-posed in the generalized sense if and only if*

$$\Omega_\alpha(\epsilon) \neq \emptyset, \quad \forall \epsilon > 0 \ \text{ and } \ \mu(\Omega_\alpha(\epsilon)) \to 0 \ \text{ as } \epsilon \to 0. \tag{4}$$

Proof Suppose that MVI(F, φ) is strongly α-well-posed in the generalized sense. Let S be the solution set of MVI(F, φ). Then S is nonempty and compact. Indeed, let $\{x_n\}$ be any sequence in S. Then $\{x_n\}$ is α-approximating for MVI(F, φ). Since MVI(F, φ) is strongly α-well-posed in the generalized sense, $\{x_n\}$ has a subsequence which converges strongly to some point of S. Thus S is compact. Clearly, $\Omega_\alpha(\epsilon) \supset S \neq \emptyset$ for all $\epsilon > 0$. Now we show that

$$\mu(\Omega_\alpha(\epsilon)) \to 0 \quad \text{as } \epsilon \to 0.$$

Observe that for every $\epsilon > 0$,

$$\mathcal{H}(\Omega_\alpha(\epsilon), S) = \max\{e(\Omega_\alpha(\epsilon), S), e(S, \Omega_\alpha(\epsilon))\} = e(\Omega_\alpha(\epsilon), S).$$

Taking into account the compactness of S, we get

$$\mu(\Omega_\alpha(\epsilon)) \leq 2\mathcal{H}(\Omega_\alpha(\epsilon), S) = 2e(\Omega_\alpha(\epsilon), S).$$

To prove (4), it is sufficient to show

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

If $e(\Omega_\alpha(\epsilon), S) \not\to 0$ as $\epsilon \to 0$, then there exist $l > 0$ and $\{\epsilon_n\} \subset R_+$ with $\epsilon_n \to 0$, and $x_n \in \Omega_\alpha(\epsilon_n)$ such that

$$x_n \notin S + B(0, l), \quad \forall n \in N, \tag{5}$$

where $B(0, l)$ is the closed ball centered at 0 with radius l. Being $x_n \in \Omega_\alpha(\epsilon_n)$, $\{x_n\}$ is an α-approximating sequence for MVI(F, φ). Since MVI(F, φ) is strongly α-well-posed in the generalized sense, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging strongly to some point of S. This contradicts to (5) and so

$$e(\Omega_\alpha(\epsilon), S) \to 0 \quad \text{as } \epsilon \to 0.$$

Conversely, assume that (4) holds. We first show that $\Omega_\alpha(\epsilon)$ is closed for all $\epsilon > 0$. Let $x_n \in \Omega_\alpha(\epsilon)$ with $x_n \to x$. Then

$$\langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \leq \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon, \quad \forall y \in H.$$

Since $z \mapsto \langle F(z), z - y\rangle$ and φ are lower semicontinuous,

$$\langle F(x), x - y\rangle + \varphi(x) - \varphi(y) \leq \frac{\alpha}{2}\|x - y\|^2 + \epsilon, \quad \forall y \in H.$$

This yields $x \in \Omega_\alpha(\epsilon)$ and so $\Omega_\alpha(\epsilon)$ is nonempty closed for all $\epsilon > 0$. Observe that

$$S = \cap_{\epsilon > 0}\Omega_\alpha(\epsilon).$$

Since $\mu(\Omega_\alpha(\epsilon)) \to 0$, the Theorem on page 412 of [12] can be applied and one concludes that S is nonempty and compact with

$$e(\Omega_\alpha(\epsilon), S) = \mathcal{H}(\Omega_\alpha(\epsilon), S) \to 0, \quad \epsilon \to 0.$$

Let $\{u_n\} \subset H$ be an α-approximating sequence for MVI(F, φ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$u_n \in dom\,\varphi, \quad \langle F(u_n), u_n - y\rangle + \varphi(u_n) - \varphi(y) \le \frac{\alpha}{2}\|u_n - y\|^2 + \epsilon_n, \quad \forall y \in H, \forall n \in N.$$

So $u_n \in \Omega_\alpha(\epsilon_n)$ follows from definition. It follows from (4) that

$$d(u_n, S) \le e(\Omega_\alpha(\epsilon_n), S) \to 0.$$

Since S is compact, there exists $\bar{x}_n \in S$ such that

$$\|u_n - \bar{x}_n\| = d(u_n, S) \to 0.$$

Again from the compactness of S, $\{\bar{x}_n\}$ has a subsequence $\{\bar{x}_{n_k}\}$ converging strongly to $\bar{x} \in S$. Hence the corresponding subsequence $\{u_{n_k}\}$ of $\{u_n\}$ converges strongly to $\bar{x}$. Thus MVI(F, φ) is strongly α-well-posed in the generalized sense. □

Now we give the following example as an application of Theorem 3.1.

Example 3.1 Let $H = R$, $F(x) = x$ and $\varphi(x) = x^2$ for all $x \in H$. Clearly, F is hemicontinuous and monotone, and φ is proper, convex and lower semicontinuous. Let $\alpha = 2$. Then

$$\begin{aligned}\Omega_2(\epsilon) &= \{x \in R : x(x-y) + x^2 - y^2 \le (x-y)^2 + \epsilon, \forall y \in R\}\\ &= \left\{x \in R : -2\left(y - \frac{x}{4}\right)^2 + \frac{9x^2}{8} - \epsilon \le 0, \forall y \in R\right\}\\ &= \left[-\frac{2\sqrt{2\epsilon}}{3}, +\frac{2\sqrt{2\epsilon}}{3}\right].\end{aligned}$$

By Theorem 3.1, MVI(F, φ) is 2-well-posed since $diam\,\Omega_2(\epsilon) = \frac{4\sqrt{2\epsilon}}{3} \to 0$ as $\epsilon \to 0$.

4 Links with well-posedness of inclusion problems

In this section we shall investigate the relations between the well-posedness of mixed variational inequalities and the well-posedness of inclusion problems. In what follows we always denote by $\to$ and $\rightharpoonup$ the strong convergence and weak convergence, respectively. Let A: $H \to 2^H$ be a set-valued mapping. The inclusion problem associated with A is defined by

$$\text{IP}(A): \quad \text{find } x \in H \text{ such that } 0 \in A(x).$$

Definition 4.1 [13,14] A sequence $\{x_n\} \subset H$ is called an approximating sequence for IP(A) if $d(0, A(x_n)) \to 0$, or equivalently, there exists $y_n \in A(x_n)$ such that $\|y_n\| \to 0$ as $n \to \infty$.

Definition 4.2 [13,14] We say that IP(A) is strongly (resp. weakly) well-posed if it has a unique solution and every approximating sequence converges strongly (resp. weakly) to the unique solution of IP(A). IP(A) is said to be strongly (resp. weakly) well-posed in the generalized sense if the solution set S of IP(A) is nonempty and every approximating sequence has a subsequence which converges strongly (resp. weakly) to a point of S.

The following theorems establish the relations between the strong (resp. weak) well-posedness of mixed variational inequalities and the strong (resp. weak) well-posedness of inclusion problems.

Theorem 4.1 *Let* $F: H \to H$ *be hemicontinuous and monotone, and let* $\varphi: H \to R \cup \{+\infty\}$ *be proper, convex and lower semicontinuous. If* MVI(F, φ) *is weakly well-posed, then* IP$(F + \partial\varphi)$ *is weakly well-posed.*

Proof Suppose that MVI(F, φ) is weakly well-posed. Then MVI(F, φ) has a unique solution x^*. By Lemma 2.1, x^* is also the unique solution of IP$(F + \partial\varphi)$. Let $\{x_n\}$ be an approximating sequence for IP$(F + \partial\varphi)$. Then there exists $y_n \in F(x_n) + \partial\varphi(x_n)$ such that $\|y_n\| \to 0$. It follows that

$$\varphi(y) - \varphi(x_n) \geq \langle y_n - F(x_n), y - x_n \rangle, \quad \forall y \in H, \forall n \in N. \tag{6}$$

If $\{x_n\}$ is unbounded, without loss of generality, we can suppose that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we can suppose that $t_n \in (0, 1]$ and $z_n \rightharpoonup z (\neq x^*)$. For any $y \in H$, it follows that

$$\begin{aligned}
&\langle F(y), z - y \rangle \\
&\quad = \langle F(y), z - z_n \rangle + \langle F(y), z_n - x^* \rangle + \langle F(y), x^* - y \rangle \\
&\quad = \langle F(y), z - z_n \rangle + t_n \langle F(y), x_n - x^* \rangle + \langle F(y), x^* - y \rangle \\
&\quad = \langle F(y), z - z_n \rangle + t_n \langle F(y), x_n - y \rangle + (1 - t_n) \langle F(y), x^* - y \rangle.
\end{aligned} \tag{7}$$

Since F is monotone,

$$\langle F(y), x^* - y \rangle \leq \langle F(x^*), x^* - y \rangle \ \text{ and } \ \langle F(y), x_n - y \rangle \leq \langle F(x_n), x_n - y \rangle. \tag{8}$$

Furthermore, we have

$$\langle F(x^*), x^* - y \rangle + \varphi(x^*) - \varphi(y) \leq 0, \quad \forall y \in H \tag{9}$$

since x^* is the unique solution of MVI(F, φ). Since φ is convex, it follows from (6) to (9) that

$$\begin{aligned}
&\langle F(y), z - y \rangle \\
&\quad \leq \langle F(y), z - z_n \rangle + t_n \varphi(y) - t_n \varphi(x_n) + t_n \langle y_n, x_n - y \rangle + (1 - t_n)[\varphi(y) - \varphi(x^*)] \\
&\quad = \langle F(y), z - z_n \rangle + \varphi(y) - [t_n \varphi(x_n) + (1 - t_n)\varphi(x^*)] + \frac{\langle y_n, x_n - y \rangle}{\|x_n - x^*\|} \\
&\quad \leq \langle F(y), z - z_n \rangle + \varphi(y) - \varphi(z_n) + \frac{\langle y_n, x_n - y \rangle}{\|x_n - x^*\|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\langle F(y), z - y \rangle \\
&\quad \leq \liminf_{n \to \infty} \left\{ \langle F(y), z - z_n \rangle + \varphi(y) - \varphi(z_n) + \frac{\langle y_n, x_n - y \rangle}{\|x_n - x^*\|} \right\} \\
&\quad \leq \varphi(y) - \varphi(z), \quad \forall y \in H.
\end{aligned}$$

This together with Lemma 2.2 yields that z solves MVI(F, φ), a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. It follows from (6) that

$$\langle F(x_{n_k}), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y) \le \langle y_{n_k}, x_{n_k} - y\rangle, \quad \forall y \in H, \forall k \in N.$$

Since F is monotone, φ is convex and lower semicontinuous, and $\|y_n\| \to 0$, we have

$$\begin{aligned}
&\langle F(y), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y)\\
&\quad \le \liminf_{k\to\infty}\{\langle F(y), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\}\\
&\quad \le \liminf_{k\to\infty}\{\langle F(x_{n_k}), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\}\\
&\quad \le \liminf_{k\to\infty}\langle y_{n_k}, x_{n_k} - y\rangle = 0, \quad \forall y \in H.
\end{aligned}$$

This together with Lemma 2.2 yields that $\bar{x}$ solves MVI(F, φ). We have $\bar{x} = x^*$ since MVI(F, φ) has a unique solution x^*. Thus x_n converges weakly to x^* and so IP$(F + \partial\varphi)$ is weakly well-posed. □

Theorem 4.2 *Let $F: H \to H$ be uniformly continuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* IP$(F + \partial\varphi)$ *is strongly (resp. weakly) well-posed, then* MVI(F, φ) *is strongly (resp. weakly) well-posed.*

Proof Let $\{x_n\}$ be an approximating sequence for MVI(F, φ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\varphi(x_n) \le \varphi(y) + \langle F(x_n), y - x_n\rangle + \epsilon_n, \quad \forall y \in H, \forall n \in N.$$

Define $\tilde{\varphi}_n : H \to R \cup \{+\infty\}$ as follows:

$$\tilde{\varphi}_n(y) = \varphi(y) + \langle F(x_n), y - x_n\rangle, \quad \forall y \in H.$$

Clearly $\tilde{\varphi}_n$ is proper, convex and lower semicontinuous and $0 \in \partial_{\epsilon_n}\tilde{\varphi}(x_n)$ for all $n \in N$. By the Brøndsted–Rockafellar theorem ([3]), there exist $\bar{x}_n \in H$ and

$$x_n^* \in \partial\tilde{\varphi}_n(\bar{x}_n) = \partial\varphi(\bar{x}_n) + F(x_n)$$

such that

$$\|x_n - \bar{x}_n\| \le \sqrt{\epsilon_n}, \quad \|x_n^*\| \le \sqrt{\epsilon_n}.$$

It follows that

$$x_n^* + F(\bar{x}_n) - F(x_n) \in (F + \partial\varphi)(\bar{x}_n).$$

Since F is uniformly continuous,

$$\|x_n^* + F(\bar{x}_n) - F(x_n)\| \le \|x_n^*\| + \|F(\bar{x}_n) - F(x_n)\| \to 0.$$

So $\{\bar{x}_n\}$ is an approximating sequence for IP$(F + \partial\varphi)$.

Let x^* be the unique solution of MVI(F, φ). By Lemma 2.1, x^* is also the unique solution of IP$(F + \partial\varphi)$.

If IP$(F + \partial\varphi)$ is strongly well-posed, then $\bar{x}_n \to x^*$. It follows that

$$\|x_n - x^*\| \le \|x_n - \bar{x}_n\| + \|\bar{x}_n - x^*\| \to 0$$

and so MVI(F, φ) is strongly well-posed.

If IP$(F + \partial\varphi)$ is weakly well-posed, then $\bar{x}_n \rightharpoonup x^*$. For any $f \in H$, we have

$$|\langle f, x_n - x^*\rangle| \leq |\langle f, x_n - \bar{x}_n\rangle| + |\langle f, \bar{x}_n - x^*\rangle| \leq \|f\|\sqrt{\epsilon_n} + |\langle f, \bar{x}_n - x^*\rangle| \to 0.$$

Thus MVI(F, φ) is weakly well-posed. □

For the well-posedness in the generalized sense, we have the following analogous results.

Theorem 4.3 *Let $F: H \to H$ be hemicontinuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* MVI(F, φ) *is strongly* (*resp. weakly*) 1*-well-posed in the generalized sense, then* IP$(F + \partial\varphi)$ *is strongly* (*resp. weakly*) *well-posed in the generalized sense.*

Proof Let $\{x_n\}$ be an approximating sequence for IP$(F + \partial\varphi)$. Then there exists $y_n \in F(x_n) + \partial\varphi(x_n)$ such that $\|y_n\| \to 0$. It follows that

$$\varphi(y) - \varphi(x_n) \geq \langle y_n - F(x_n), y - x_n\rangle, \quad \forall y \in H, \forall n \in N$$

and so

$$\langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \leq \langle y_n, x_n - y\rangle \leq \frac{1}{2}\|x_n - y\|^2 + \frac{1}{2}\|y_n\|^2, \quad \forall y \in H, \forall n \in N.$$

This together with $\|y_n\| \to 0$ implies that $\{x_n\}$ is 1-approximating for MVI(F, φ). Since MVI(F, φ) is strongly (resp. weakly) 1-well-posed in the generalized sense, x_n converges strongly (resp. weakly) to some solution x^* of MVI(F, φ). By Lemma 2.1, x^* is also a solution of IP$(F + \partial\varphi)$. So IP$(F + \partial\varphi)$ is strongly (resp. weakly) well-posed in the generalized sense. □

Theorem 4.4 *Let $F: H \to H$ be uniformly continuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* IP$(F + \partial\varphi)$ *is strongly* (*resp. weakly*) *well-posed in the generalized sense, then* MVI(F, φ) *is strongly* (*resp. weakly*) *well-posed in the generalized sense.*

Proof The conclusion follows from similar arguments as Theorem 4.2. □

5 Links with well-posedness of fixed point problems

In this section we shall investigate the relations between the well-posedness of mixed variational inequalities and the well-posedness of fixed point problems. Let $T: H \to H$ be a single-valued mapping. The fixed-point problem associated with T is defined by

$$\text{FP}(T): \quad \text{find } x \in H \text{ such that } T(x) = x.$$

We first recall some concepts.

Definition 5.1 [13,14] A sequence $\{x_n\} \subset H$ is called an approximating sequence for FP(T) if $\|x_n - T(x_n)\| \to 0$ as $n \to \infty$.

Definition 5.2 [13,14] We say that FP(T) is strongly (resp. weakly) well-posed if FP(T) has a unique solution and every approximating sequence for FP(T) converges strongly (resp. weakly) to the unique solution. FP(T) is said to be strongly (resp. weakly) well-posed in the generalized sense if FP(T) has a nonempty solution set S and every approximating sequence for FP(T) has a subsequence which converges strongly (resp. weakly) to some point of S.

Theorem 5.1 *Let $F: H \to H$ be uniformly continuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* MVI(F, φ) *is weakly well-posed, then* FP$(J_\varphi^\lambda(I - \lambda F))$ *is weakly well-posed, where $\lambda > 0$ is a constant.*

Proof Suppose that MVI(F, φ) is weakly well-posed. Let x^* be the unique solution of MVI(F, φ). By Lemma 2.1, x^* is also the unique solution of FP$(J_\varphi^\lambda(I - \lambda F))$. Let $\{x_n\}$ be an approximating sequence for FP$(J_\varphi^\lambda(I - \lambda F))$. Then $\|x_n - w_n\| \to 0$, where

$$w_n = J_\varphi^\lambda(I - \lambda F)(x_n) = J_\varphi^\lambda(x_n - \lambda F(x_n)).$$

By the definition of J_φ^λ,

$$\frac{x_n - w_n}{\lambda} - F(x_n) \in \partial\varphi(w_n).$$

It follows that

$$\varphi(y) - \varphi(w_n) \geq \left\langle \frac{x_n - w_n}{\lambda} - F(x_n), y - w_n \right\rangle, \quad \forall y \in H, \forall n \in N. \tag{10}$$

If $\{w_n\}$ is unbounded, without loss of generality, we can suppose that $\|w_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|w_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we can suppose that $t_n \in (0, 1]$ and $z_n \rightharpoonup z(\neq x^*)$. By similar arguments as Theorem 4.1, we have

$$\begin{aligned}
&\langle F(y), z - y\rangle \\
&\quad \leq \langle F(y), z - z_n\rangle + t_n\langle F(w_n) - F(x_n), w_n - y\rangle \\
&\qquad + \varphi(y) - \varphi(z_n) + \frac{t_n}{\lambda}\langle w_n - x_n, y - w_n\rangle, \quad \forall y \in H, \forall n \in N.
\end{aligned}$$

Since F is uniformly continuous, φ is convex and lower semicontinuous, it follows that

$$\begin{aligned}
&\langle F(y), z - y\rangle \\
&\quad \leq \liminf_{n\to\infty} \Big\{ \langle F(y), z - z_n\rangle + t_n\langle F(w_n) - F(x_n), w_n - y\rangle \\
&\qquad + \varphi(y) - \varphi(z_n) + \frac{t_n}{\lambda}\langle w_n - x_n, y - w_n\rangle \Big\} \\
&\quad \leq \varphi(y) - \varphi(z), \quad \forall y \in H.
\end{aligned}$$

This together with Lemma 2.2 implies that z solves MVI(F, φ), a contradiction. Thus, $\{w_n\}$ is bounded.

Let $\{w_{n_k}\}$ be any subsequence of $\{w_n\}$ such that $w_{n_k} \rightharpoonup \bar{w}$ as $k \to \infty$. From (10), we have

$$\begin{aligned}
&\langle F(w_{n_k}), w_{n_k} - y\rangle + \varphi(w_{n_k}) - \varphi(y) \\
&\quad \leq \left\langle \frac{x_{n_k} - w_{n_k}}{\lambda}, w_{n_k} - y \right\rangle + \langle F(w_{n_k}) - F(x_{n_k}), w_{n_k} - y\rangle, \quad \forall y \in H.
\end{aligned}$$

Since F is monotone and uniformly continuous, and φ is convex and lower semicontinuous,

$$\begin{aligned}
&\langle F(y), \bar{w} - y\rangle + \varphi(\bar{w}) - \varphi(y) \\
&\quad \le \liminf_{k\to\infty}\{\langle F(y), w_{n_k} - y\rangle + \varphi(w_{n_k}) - \varphi(y)\} \\
&\quad \le \liminf_{k\to\infty}\{\langle F(w_{n_k}), w_{n_k} - y\rangle + \varphi(w_{n_k}) - \varphi(y)\} \\
&\quad \le \liminf_{k\to\infty}\left\{\left\langle \frac{x_{n_k} - w_{n_k}}{\lambda}, w_{n_k} - y\right\rangle + \langle F(w_{n_k}) - F(x_{n_k}), w_{n_k} - y\rangle\right\} \\
&\quad = 0, \quad \forall y \in H, \forall n \in N.
\end{aligned}$$

This together with Lemma 2.2 yields that $\bar{w}$ solves MVI(F, φ). We have $w_n \rightharpoonup x^*$ since MVI(F, φ) has a unique solution x^*. For any $f \in H$, it follows that

$$\begin{aligned}
|\langle f, x_n - x^*\rangle| &\le |\langle f, x_n - w_n\rangle| + |\langle f, w_n - x^*\rangle| \\
&\le \|f\| \cdot \|x_n - w_n\| + |\langle f, w_n - x^*\rangle| \to 0.
\end{aligned}$$

Thus $x_n \rightharpoonup x^*$ and so FP$(J_\varphi^\lambda(I - \lambda F))$ is weakly well-posed. □

Theorem 5.2 *Let $F\colon H \to H$ be uniformly continuous and monotone, and let $\varphi\colon H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* FP$(J_\varphi^\lambda(I - \lambda F))$ *is strongly (resp. weakly) well-posed, then* MVI(F, φ) *is strongly (resp. weakly) well-posed.*

Proof Let $\{x_n\}$ be an approximating sequence for MVI(F, φ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\varphi(x_n) \le \varphi(y) + \langle F(x_n), y - x_n\rangle + \epsilon_n, \quad \forall y \in H, \forall n \in N.$$

Define $\tilde{\varphi}_n\colon H \to R \cup \{+\infty\}$ as follows:

$$\tilde{\varphi}_n(y) = \varphi(y) + \langle F(x_n), y - x_n\rangle, \quad \forall y \in H.$$

Clearly $\tilde{\varphi}_n$ is proper, convex and lower semicontinuous and $0 \in \partial_{\epsilon_n}\tilde{\varphi}(x_n)$ for all $n \in N$. By the Brøndsted–Rockafellar theorem ([3]), there exist $\bar{x}_n \in H$ and

$$x_n^* \in \partial\tilde{\varphi}_n(\bar{x}_n) = \partial\varphi(\bar{x}_n) + F(x_n) \tag{11}$$

such that

$$\|x_n - \bar{x}_n\| \le \sqrt{\epsilon_n}, \quad \|x_n^*\| \le \sqrt{\epsilon_n}. \tag{12}$$

From (11), we have

$$\bar{x}_n = J_\varphi^\lambda[\bar{x}_n + \lambda x_n^* - \lambda F(x_n)]. \tag{13}$$

It follows from (12) to (13) that

$$\begin{aligned}
&\|\bar{x}_n - J_\varphi^\lambda(I - \lambda F)(\bar{x}_n)\| \\
&\quad = \|J_\varphi^\lambda[\bar{x}_n + \lambda x_n^* - \lambda F(x_n)] - J_\varphi^\lambda[\bar{x}_n - \lambda F(\bar{x}_n)]\| \\
&\quad \le \|\lambda x_n^* + \lambda[F(\bar{x}_n) - F(x_n)]\| \\
&\quad \le \lambda\|x_n^*\| + \lambda\|F(\bar{x}_n) - F(x_n)\| \to 0
\end{aligned}$$

and so $\{\bar{x}_n\}$ is an approximating sequence for FP$(J_\varphi^\lambda(I - \lambda F))$.

Let x^* be the unique solution of FP$(J_\varphi^\lambda(I - \lambda F))$. By Lemma 2.1, x^* is also the unique solution of MVI(F, φ).

If $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ is strongly well-posed, then $\bar{x}_n \to x^*$. It follows that

$$\|x_n - x^*\| \le \|x_n - \bar{x}_n\| + \|\bar{x}_n - x^*\| \to 0.$$

Thus $\mathrm{MVI}(F, \varphi)$ is strongly well-posed.

If $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ is weakly well-posed, then $\bar{x}_n \rightharpoonup x^*$. For any $f \in H$, we have

$$\begin{aligned}|\langle f, x_n - x^*\rangle| &\le |\langle f, x_n - \bar{x}_n\rangle| + |\langle f, \bar{x}_n - x^*\rangle| \\ &\le \|f\|\sqrt{\epsilon_n} + |\langle f, \bar{x}_n - x^*\rangle| \to 0\end{aligned}$$

and so $\mathrm{MVI}(F, \varphi)$ is weakly well-posed. □

For the well-posedness in the generalized sense, we have the following result.

Theorem 5.3 *Let $F: H \to H$ be uniformly continuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* $\mathrm{MVI}(F, \varphi)$ *is strongly* (*resp. weakly*) $(1+\frac{1}{\lambda})$*-well-posed in the generalized sense, then* $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ *is strongly* (*resp. weakly*) *well-posed in the generalized sense, where $\lambda > 0$ is a constant.*

Proof Let $\{x_n\}$ be an approximating sequence for $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$. Then $\|x_n - w_n\| \to 0$, where

$$w_n = J_\varphi^\lambda(I-\lambda F)(x_n) = J_\varphi^\lambda(x_n - \lambda F(x_n)).$$

By the definition of J_φ^λ,

$$\frac{x_n - w_n}{\lambda} - F(x_n) \in \partial\varphi(w_n).$$

From the definition of subdifferential, we get

$$\varphi(y) - \varphi(w_n) \ge \left\langle \frac{x_n - w_n}{\lambda} - F(x_n), y - w_n \right\rangle, \quad \forall y \in H, \forall n \in N.$$

It follows that

$$\begin{aligned}&\langle F(w_n), w_n - y\rangle + \varphi(w_n) - \varphi(y) \\ &\quad \le F(w_n) - F(x_n), w_n - y\rangle + \frac{1}{\lambda}\langle x_n - w_n, w_n - y\rangle \\ &\quad \le \frac{1}{2}(1+\frac{1}{\lambda})\|w_n - y\|^2 + \left(\frac{1}{2}\|F(w_n) - F(x_n)\|^2 + \frac{1}{2\lambda}\|x_n - w_n\|^2\right), \quad \forall y \in H, \forall n \in N.\end{aligned}$$

Since F is uniformly continuous and $\|w_n - x_n\| \to 0$, $\{w_n\}$ is $(1+\frac{1}{\lambda})$-approximating for $\mathrm{MVI}(F, \varphi)$.

If $\mathrm{MVI}(F, \varphi)$ is strongly $(1+\frac{1}{\lambda})$-well-posed in the generalized sense, then $\{w_n\}$ has a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \to \bar{x}^*$ as $k \to \infty$, where x^* is a solution of $\mathrm{MVI}(F, \varphi)$. By Lemma 2.1, x^* is also a solution of $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$. It follows that

$$\|x_{n_k} - x^*\| \le \|x_{n_k} - w_{n_k}\| + \|w_{n_k} - x^*\| \to 0$$

as $k \to \infty$. Thus $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ is strongly well-posed in the generalized sense.

If $\mathrm{MVI}(F, \varphi)$ is weakly $(1+\frac{1}{\lambda})$-well-posed in the generalized sense, then $\{w_n\}$ has a subsequence $\{w_{n_k}\}$ such that $w_{n_k} \rightharpoonup \bar{x}^*$ as $k \to \infty$, where x^* is a solution of $\mathrm{MVI}(F, \varphi)$.

By Lemma 2.1, x^* is also a solution of $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$. For any $f \in H$, it follows that, as $k \to \infty$,

$$\begin{aligned}|\langle f, x_{n_k} - x^*\rangle| &\le |\langle f, x_{n_k} - w_{n_k}\rangle| + |\langle f, w_{n_k} - x^*\rangle| \\ &\le \|f\| \cdot \|x_{n_k} - w_{n_k}\| + |\langle f, w_{n_k} - x^*\rangle| \to 0.\end{aligned}$$

Thus $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ is weakly well-posed in the generalized sense. □

Theorem 5.4 *Let $F: H \to H$ be uniformly continuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If* $\mathrm{FP}(J_\varphi^\lambda(I-\lambda F))$ *is strongly (resp. weakly) well-posed in the generalized sense, then* $\mathrm{MVI}(F, \varphi)$ *is strongly (resp. weakly) well-posed in the generalized sense.*

Proof The conclusion follows from similar arguments as Theorem 5.2. □

6 Conditions for well-posedness

In this section we shall prove that under suitable conditions the well-posedness of the mixed variational inequality is equivalent to the existence and uniqueness of its solutions, and the well-posedness in the generalized sense is equivalent to the existence of its solutions.

Theorem 6.1 *Let $F: H \to H$ be hemicontinuous and monotone, and let $\varphi: H \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. Then,* $\mathrm{MVI}(F, \varphi)$ *is weakly well-posed if and only if it has a unique solution.*

Proof The necessity is obvious. For the sufficiency, suppose that $\mathrm{MVI}(F, \varphi)$ has a unique solution x^*. If $\mathrm{MVI}(F, \varphi)$ is not weakly well-posed, then there exists an approximating sequence $\{x_n\}$ for $\mathrm{MVI}(F, \varphi)$ such that $x_n \not\rightharpoonup x^*$. Thus, there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \le \epsilon_n, \quad \forall y \in H, \forall n \in N. \tag{14}$$

If $\{x_n\}$ is unbounded, without loss of generality, we can suppose that $\|x_n\| \to +\infty$. Let

$$t_n = \frac{1}{\|x_n - x^*\|}, \quad z_n = x^* + t_n(x_n - x^*).$$

Without loss of generality, we can suppose that $t_n \in (0, 1]$ and $z_n \rightharpoonup z (\ne x^*)$. By similar arguments as in Theorem 4.1, we have

$$\langle F(y), z - y\rangle \le \langle F(y), z - z_n\rangle + \varphi(y) - \varphi(z_n) + t_n\epsilon_n, \quad \forall y \in H, \quad \forall n \in N.$$

It follows that

$$\begin{aligned}&\langle F(y), z - y\rangle \\ &\quad \le \liminf_{n\to\infty}\{\langle F(y), z - z_n\rangle + \varphi(y) - \varphi(z_n) + t_n\epsilon_n\} \\ &\quad \le \varphi(y) - \varphi(z), \quad \forall y \in H.\end{aligned}$$

This together with Lemma 2.2 yields that z solves $\mathrm{MVI}(F, \varphi)$, a contradiction. Thus, $\{x_n\}$ is bounded.

Let $\{x_{n_k}\}$ be any subsequence of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \bar{x}$ as $k \to \infty$. It follows from (14) that

$$\langle F(x_{n_k}), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y) \le \epsilon_{n_k}, \quad \forall y \in H, \quad \forall n \in N.$$

Since F is monotone and φ is convex and lower semicontinuous, we have

$$\begin{aligned}
&\langle F(y), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \\
&\quad \le \liminf_{k\to\infty}\{\langle F(y), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\} \\
&\quad \le \liminf_{k\to\infty}\{\langle F(x_{n_k}), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\} \\
&\quad \le \liminf_{k\to\infty} \epsilon_{n_k} = 0, \quad \forall y \in H.
\end{aligned}$$

This together with Lemma 2.2 yields that $\bar{x}$ solves MVI(F, φ). We have $\bar{x} = x^*$ since MVI(F, φ) has a unique solution x^*. Thus x_n converges weakly to x^*, a contradiction. So MVI(F, φ) is weakly well-posed. □

Example 6.1 Let H, F, φ be defined as in Example 3.1. It is easy to see that F is hemicontnuous and monotone, φ is proper, convex and lower semicontinuous, and MVI(F, φ) has a unique solution $x^* = 0$. By Theorem 6.1, MVI(F, φ) is well-posed.

Theorem 6.2 *Let $F: R^m \to R^m$ be hemicontinuous and monotone, and let $\varphi: R^m \to R \cup \{+\infty\}$ be proper, convex and lower semicontinuous. If there exists some $\epsilon > 0$ such that $\Omega_\alpha(\epsilon)$ is nonempty bounded, then* MVI(F, φ) *is α-well-posed in the generalized sense.*

Proof Let $\{x_n\}$ be an α-approximating sequence for MVI(F, φ). Then there exists $\epsilon_n > 0$ with $\epsilon_n \to 0$ such that

$$\langle F(x_n), x_n - y\rangle + \varphi(x_n) - \varphi(y) \le \frac{\alpha}{2}\|x_n - y\|^2 + \epsilon_n, \quad \forall y \in R^m, \forall n \in N. \tag{15}$$

Let $\epsilon > 0$ be such that $\Omega_\alpha(\epsilon)$ is nonempty bounded. Then there exists n_0 such that $x_n \in \Omega_\alpha(\epsilon)$ for all $n > n_0$. This implies that $\{x_n\}$ is bounded and so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \bar{x}$ as $k \to \infty$. Since F is monotone and φ is convex and lower semicontinuous, it follows from (15) that

$$\begin{aligned}
&\langle F(y), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \\
&\quad \le \liminf_{k\to\infty}\{\langle F(y), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\}) \\
&\quad \le \liminf_{k\to\infty}\{\langle F(x_{n_k}), x_{n_k} - y\rangle + \varphi(x_{n_k}) - \varphi(y)\} \\
&\quad \le \liminf_{k\to\infty}\left\{\frac{\alpha}{2}\|x_{n_k} - y\|^2 + \epsilon_{n_k}\right\} \\
&\quad = \frac{\alpha}{2}\|\bar{x} - y\|^2, \quad \forall y \in R^m.
\end{aligned}$$

For any $y \in R^m$, let $y(t) = \bar{x} + t(y - \bar{x})$ for all $t \in (0, 1)$. Then

$$\langle F(y(t)), \bar{x} - y(t)\rangle + \varphi(\bar{x}) - \varphi(y(t)) \le \frac{\alpha}{2}\|\bar{x} - y(t)\|^2.$$

By the convexity of φ,

$$\langle F(y(t)), \bar{x} - y\rangle + \varphi(\bar{x}) - \varphi(y) \le \frac{t\alpha}{2}\|\bar{x} - y\|^2, \quad \forall y \in R^m, \quad \forall t \in (0, 1).$$

Letting $t \to 0$ in the above inequality, we have

$$\langle F(y), \bar{x} - y \rangle + \varphi(\bar{x}) - \varphi(y) \le 0, \quad \forall y \in R^m,$$

which together with Lemma 2.2 implies that $\bar{x}$ solves MVI(F, φ). Thus MVI(F, φ) is well-posed in the generalized sense. □

Theorem 6.2 says nothing but that, under suitable conditions, the α-well-posedness in the generalized sense is equivalent to the existence of solutions.

The following example shows the assumption that $\Omega_\alpha(\epsilon)$ is nonempty bounded for some $\epsilon > 0$ is essential in Theorem 6.2.

Example 6.2 Let $m = 1$, $F(x) = 0$, and $\varphi(x) = \delta_K(x)$, where $K = [0, +\infty)$. Clearly, F is hemicontnuous and monotone, and φ is proper, convex and lower semicontinuous. For any $\epsilon > 0$, we have $\Omega_\alpha(\epsilon) = [0, +\infty)$. By Theorem 3.2, MVI($F, \varphi$) is not α-well-posed in the generalized sense.

7 Conclusions

In this paper we introduce some concepts of well-posedness for mixed variational inequalities. In Sect. 3, we establish some metric characterizations of strong α-well-posedness. In Sect. 4, we discuss the connections between the strong (weak) well-posedness of mixed variational inequalities and strong (weak) well-posedness of inclusion problems. In Sect. 5, we further investigate the relationships between the strong (weak) well-posedness of mixed variational inequalities and the strong (weak) well-posedness of fixed point problems. In Sect. 6, we prove that under suitable conditions, the well-posedness of a mixed variational inequalities is the existence and uniqueness of solutions, and that the well-posedness in the generalized sense is equivalent to the existence of solutions.

It is known that the concept of α-well-posedness has been introduced for optimization problems [5], variational inequalities [5,17] and Nash equilibrium problems [17]. Now one open problem arises in a natural way:

(a) Is it possible to consider the concept of α-well-posedness for the inclusion problems? In Theorems 3.1–3.2, we give some characterizations of strong well-posedness. Another open problem is as follows:
(b) Is it possible to give a metric characterization only for weak well-posedness?

As pointed out by a referee, it is deserved to consider the above two open problems (a) and (b) in the future.

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